Set theory Miscellany

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ABSTRACT: A NaPo conundrum. Meir's Finite intersection problem. Pre-image problem.
`Unfinished: Ordinal arithmetic. Cardinal arithmetic. Not yet: Hercules and the Hydra.`

Notation. Two sets P and Q are ***equinumerous***, or "***bijective*** with each other", if *there exists* a bijection $P \hookrightarrow\!\!\!\rightarrow Q$. [BTWay, we use a hook-arrow to indicate an injection, e.g, $P \hookrightarrow Q$, and a doublehead-arrow, e.g $P \twoheadrightarrow Q$ to indicate a surjection. Hence $\hookrightarrow\!\!\!\rightarrow$ indicates a bijection.] Write the equinumerous relation as

$$P \asymp Q\,.$$

Write $P \preccurlyeq Q$ if *there exists* an injection $P \hookrightarrow Q$. Finally, let $P \prec Q$ mean that $P \preccurlyeq Q$ yet $P \not\asymp Q$.

Easily, $\asymp$ is an equivalence relation. [On the class of cardinalities, relation $\preccurlyeq$ is a pre-order. Is $\preccurlyeq$ a *partial-order*? Is $\preccurlyeq$ a *total-order*?]

Call S ***countably-infinite*** or ***denumerable*** if $S \asymp \mathbb{N}$. Set S is ***countable*** if $S \preccurlyeq \mathbb{N}$, i.e, S is bijective with some subset of $\mathbb{N}$. [So a countable set is either *finite* or *countably-infinite*.]

Problem V4 from a NaPo exam

Below, A,P,Q are arbitrary sets.

Soln-**V4a:** Here is a bijection $\mathbf{H}\colon A^{P\times Q} \hookrightarrow\!\!\!\rightarrow [A^P]^Q$. Let $\mathbf{H}(f) := \widehat{f}$, where $\widehat{f}(q)(p) := f(p,q)$. Or in one swell foop,

$$\mathbf{H}(f) := [q \mapsto [p \mapsto f(p,q)]]\,.$$

b Below, I need some two 2-element sets; let's take $\{\mathbf{1},\mathbf{2}\}$ and $\{\heartsuit,\clubsuit\}$. Use "2^B" to abbreviate $\{\heartsuit,\clubsuit\}^B$; the set of maps from $B \to \{\heartsuit,\clubsuit\}$.

I am given bijections

$$\mathcal{P}\colon\mathbb{N} \hookrightarrow\!\!\!\rightarrow \mathbb{N}\times\{\mathbf{1},\mathbf{2}\} \quad\text{and}\quad \mathcal{Q}\colon\mathbb{R} \hookrightarrow\!\!\!\rightarrow 2^{\mathbb{N}}\,,$$

and the inverse-fnc $\mathcal{G} := \mathcal{Q}^{-1}$ mapping $2^{\mathbb{N}} \hookrightarrow\!\!\!\rightarrow \mathbb{R}$. **ITOf** $\mathcal{P},\mathcal{Q},\mathcal{G}$, I want to define these bijections:

†1: $\beta\colon\mathbb{R}^2 \hookrightarrow\!\!\!\rightarrow 2^{\mathbb{N}}\times 2^{\mathbb{N}}\,.$

†2: $\gamma\colon 2^{\mathbb{N}}\times 2^{\mathbb{N}} \hookrightarrow\!\!\!\rightarrow 2^{\mathbb{N}\times\{\mathbf{1},\mathbf{2}\}}\,.$

†3: $\delta\colon 2^{\mathbb{N}\times\{\mathbf{1},\mathbf{2}\}} \hookrightarrow\!\!\!\rightarrow 2^{\mathbb{N}}\,.$

I'll then combine them to produce a bijection

†4: $\varepsilon\colon\mathbb{R}^2 \hookrightarrow\!\!\!\rightarrow \mathbb{R}\,.$

Soln-**V4b:** Well, $\mathbb{R}^2$ is $\mathbb{R}\times\mathbb{R}$, so it works to define

‡1: $\beta(x_{\mathbf{1}},x_{\mathbf{2}}) := \big(\mathcal{Q}(x_{\mathbf{1}}),\mathcal{Q}(x_{\mathbf{2}})\big)\,.$

Using f for a general element of $2^{\mathbb{N}}$, we can write (f_1,f_2) for a gen-elt of $2^{\mathbb{N}}\times 2^{\mathbb{N}}$. Employing $\mathbf{j}\in\{\mathbf{1},\mathbf{2}\}$ as an index variable, let

‡2: $\gamma(f_{\mathbf{1}},f_{\mathbf{2}}) := \Big[(n,\mathbf{j}) \mapsto f_{\mathbf{j}}(n)\Big]\,.$

For m a natnum, the value $\mathcal{P}(m)$ is a natnum-index pair. Write this pair as $(m_{\mathbb{N}}, m_{\mathbf{Idx}})$. For example:

> *When* $\mathcal{P}(5177) = (38,\mathbf{2})$, *then* $5177_{\mathbb{N}} = 38$ *and* $5177_{\mathbf{Idx}} = \mathbf{2}$.

Use h for a general fnc in $2^{\mathbb{N}\times\{\mathbf{1},\mathbf{2}\}}$. Let

‡3: $$\delta(h) := [m \mapsto h(m_{\mathbb{N}}, m_{\mathbf{Idx}})] \overset{\text{note}}{=\!=} h\circ\mathcal{P}\,.$$

This δ is indeed a bijection $2^{\mathbb{N}\times\{\mathbf{1},\mathbf{2}\}} \hookrightarrow\!\!\!\rightarrow 2^{\mathbb{N}}$.

Composing functions. Here is $\gamma\circ\beta$ written out in full:

$$\gamma(\beta(x_{\mathbf{1}},x_{\mathbf{2}})) = \Big[(n,\mathbf{j}) \mapsto \mathcal{Q}(x_{\mathbf{j}})(n)\Big]\,.$$

So $\delta\circ\gamma\circ\beta$ maps $(x_{\mathbf{1}},x_{\mathbf{2}})$ to $[m \mapsto \mathcal{Q}(x_{m_{\mathbf{Idx}}})(m_{\mathbb{N}})]$.

Defining $\varepsilon := \mathcal{G}\circ\delta\circ\gamma\circ\beta$ produces

$$\varepsilon(x_{\mathbf{1}},x_{\mathbf{2}}) = \mathcal{G}\Big(m \mapsto \mathcal{Q}(x_{m_{\mathbf{Idx}}})(m_{\mathbb{N}})\Big)\,.$$

Intersection Problems

[JK: Tuesday, 12Jun2012] T.fol question arose when trying to figure out why the countable Marriage-lemma didn't generalize in the obvious way to higher cardinalities.

Say that a set S is ***sequentially disappearing*** (***SD***) if *there exists* a sequence of sets

$$S \supset D_1 \supset D_2 \supset \ldots\,, \quad \textit{with} \quad \Big[\bigcap_{j=1}^{\infty} D_j\Big] = \varnothing$$

and each $|D_j| = |S|$.

Sierpinski's question

Consider a space X. Suppose P is a cardinality property such as "finite", "infinite", or "co-finite".

Define a corresponding property of a family $\mathcal{C}$ of subsets of X. Say that $\mathcal{C}$ is ***P-paired*** (on X) if whenever $A, B \in \mathcal{C}$ are distinct, then $A \cap B$ has property P.

1: Sierpinski's query. *Suppose $\mathcal{C}$ is a finite-paired family on $\mathbb{N}$. Can $\mathcal{C}$ have the cardinality of $\mathbb{R}$?* ◊

Pf of ***Yes*** *(Sierpinski).* Let $\varphi : \mathbb{Q} \hookrightarrow\!\!\!\to \mathbb{N}$ be a bijection. Use the Axiom of Choice to pick, for each $x \in \mathbb{R}$, a sequence $(q_n)_{n=1}^{\infty}$ of rationals which converges to x. Define A_x to be $\{\varphi(q_n)\}_{n=1}^{\infty}$. Evidently if $x \neq y$ then $A_x \cap A_y$ is finite. Thus $\mathcal{C} := \{A_x\}_{x \in \mathbb{R}}$ is finite-paired. ♦

Proof of ***Yes*** *(Smorodinsky).* Label the squares of the $\mathbb{Z} \times \mathbb{Z}$ lattice with $\mathbb{N}$, bijectively; these squares are closed on the bottom and left, and open on the top and right. For each angle θ, let A_θ consist of the numbers of all squares through which passes the ray (from the origin) at angle θ. ♦

2: Meir's question. *Suppose family $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ is [at most 8]-paired. Can $\mathcal{C}$ be uncountable?* ◊

NB. For both proofs below: WLOGenerality, $\mathcal{C}$ contains only infinite sets. (There are but countably many finite sets of naturals.) □

Pf of ***No*** *(jk).* Fix a cardinality-8 subset $F \subset \mathbb{N}$. Let $\mathcal{D}_F$ consist of those members $A \in \mathcal{C}$ such that $A \supset F$. Imagine deleting F from $\mathbb{N}$. This does not identify any two members of $\mathcal{D}_F$, but *now* the members of $\mathcal{D}_F$ are pairwise disjoint. And now it is evident that $\mathcal{D}_F$ is countable, since its elements union-disjointly to a countable set.

Finally, since $\mathcal{C}$ is the countable union $\bigcup_F \mathcal{D}_F$, as F ranges over all cardinality-8 sets, $\mathcal{C}$ is seen to be countable. (Note that $\bigcup_F \mathcal{D}_F$ need not be a *disjoint* union.) ♦

Proof of ***No*** *(Kyle Duffy).* Define a fnc $f : \mathcal{C} \to \mathbb{N}^{\times 9}$ by $f(A)$ is the 9-tuple of the 9 smallest elements of A, in order. This f is injective, thanks to the [at most 8]-paired property. ♦

3: Someone's question. *Let $\mathcal{P}_\infty$ be the collection of all infinite subsets of $\mathbb{N}$. Does there exist a map $f : \mathcal{P}_\infty \to \mathbb{N}$ such that for each $y \in \mathbb{N}$, the pre-image $f^{-1}(y)$ has this property:*

†: *Each finitely many members of $f^{-1}(y)$ has infinite-intersection.* ◊

Pf of ***No*** *(jk).* Even weakening (†) to

‡: *Each two elts of $f^{-1}(y)$ has ∞-intersection*

does not permit such an f. For using the answer to (1), let $\mathcal{C}$ be a finite-paired family on $\mathbb{N}$, with $\mathcal{C} \asymp \mathbb{R}$. Then (‡) forces restriction $f|_{\mathcal{C}}$ to be injective. So $\mathcal{C} \preccurlyeq \text{Range}(f)$, i.e, $\mathbb{R} \preccurlyeq \mathbb{N}$; ⋇ ♦

§A Ordinals and Cardinals

Ordinal arithmetic

One defn of ***ordinal*** is a well-order type. That is, a class of well-ordered sets which are order-isomorphic to each other.

Cardinal arithmetic

First, a lemma on sets. Use $\asymp$ to mean "bijective with", when applied to sets, and mean "equal cardinality", when applied to cardinals.

4: Theorem. *Here ν, β, κ are cardinals.*

a: *A bijection from $[S^P]^A$ to $S^{A\times P}$ is $f \mapsto \widehat{f}$ where $\widehat{f}(a,p) := [f(a)](p)$.*

b: *If at least one of β and κ is infinite, then*

$$\beta + \kappa \asymp \operatorname{Max}(\beta, \kappa).$$

c: *If neither β nor κ is zero, and at least one is infinite, then*

$$\beta \times \kappa \asymp \operatorname{Max}(\beta, \kappa).$$

d: *If β is infinite and $2 \preccurlyeq \kappa \preccurlyeq 2^\beta$, then $\kappa^\beta \asymp 2^\beta$.* ♢

Pf of (b). WLOG $\beta \le \kappa$. Note $\kappa \preccurlyeq \beta + \kappa \preccurlyeq 2\kappa$. Schröder-Bernstein will finish the argument if we can establish $2\kappa \preccurlyeq \kappa$, once κ is infinite. Recall that $\omega + 1 \asymp \omega$ and, since ω is an initial segment of every infinite ordinal,

$$\sigma \text{ infinite} \implies \sigma{+}1 \asymp \sigma.$$

Cantor diagonalization proves $2\kappa \asymp \kappa$, when $\kappa \asymp \omega$. The successor case $\kappa = \sigma + 1$ is simply the computation

$$2\kappa \asymp 2\sigma + 2 \asymp \sigma + 2 \asymp \sigma + 1 \asymp \kappa.$$

As for the limit case, let β range over all ordinals less than κ. Fix an $\alpha < \kappa$ and compute
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$$\kappa \asymp \sup_\beta \beta \asymp \sup_\beta 2\beta \ge 2\alpha.$$

Taking a supremum over α ♦

Now use transfinite induction.

Successor case, $\alpha = \beta + 1$:

This β must be infinite and so

$$\alpha \asymp \beta \asymp \omega{\times}\beta \asymp \omega{\times}\alpha.$$

α is a limit ordinal:

When $\alpha = \beta + 1$, then β is infinite and

$$\alpha \asymp \omega{\times}\alpha.$$

Proof of (d). Note $\kappa^1 \le \kappa^\beta \le [2^\beta]^\beta = 2^{\beta\beta}$ which, since β is infinite, equals 2^β. Thus $\kappa \le \kappa^\beta \le \kappa$. ♦

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